

A LOCAL FORM FOR THE AUTOMORPHISMS OF THE SPECTRAL UNIT BALL

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ABSTRACT. If F is an automorphism of Ω_n , the n^2 -dimensional spectral unit ball, we show that, in a neighborhood of any cyclic matrix of Ω_n , the map F can be written as conjugation by a holomorphically varying non singular matrix. This provides a shorter proof of a theorem of J. Rostand, with a slightly stronger result.

1. BACKGROUND

Let \mathcal{M}_n be the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$ denote by $sp(A)$ the spectrum of A . The spectral ball Ω_n is the set

$$\Omega_n := \{A \in \mathcal{M}_n : \forall \lambda \in sp(A), |\lambda| < 1\}.$$

Let F be an automorphism of Ω_n , that is to say, a biholomorphic map of the spectral ball into itself. Ransford and White [6] proved that, by composing with a natural lifting of a Möbius map of the disk, one could reduce oneself to the case where $F(0) = 0$, and that in that case the linear map $F'(0)$ was a linear automorphism of Ω_n , so that by composing with its inverse, one is reduced to the case $F(0) = 0$, $F'(0) = I$ (the identity map). We then say that the automorphism is *normalized*. Ransford and White [6] proved that such automorphisms preserve the spectrum of matrices.

We say that two matrices X, Y are *conjugate* if there exists $Q \in \mathcal{M}_n^{-1}$ such that $X = Q^{-1}YQ$.

Baribeau and Ransford [1] (see also [2] for a more elementary proof) proved that every spectrum-preserving \mathcal{C}^1 -diffeomorphism of an open subset of \mathcal{M}_n , and thus every normalized automorphism of the spectral ball is a pointwise conjugation:

$$(1) \quad F(X) = Q(X)^{-1}XQ(X).$$

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Rostand's contribution [7] was to show that $Q(X)$ could be chosen locally holomorphically in a neighborhood of every X admitting n distinct eigenvalues.

We will give a short proof of a slightly stronger result: the exceptional set of matrices where the local holomorphic choice cannot be guaranteed will be of complex codimension 2 instead of 1.

The motivation for this result was a conjecture formulated in [6] about the automorphisms of the spectral ball, which reduces to asking whether any normalized automorphisms can be written in the form (1), where Q would be globally homomorphic on Ω_n , and depend only on the conjugacy class of X . Notice that a recent result of Zwonek [8] shows that any proper map of the spectral ball to itself is actually an automorphism of it, so that the proof of the Ransford-White conjecture would yield a description of all the proper maps of the spectral ball into itself.

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2. STATEMENT

Definition 1. *We say that a matrix M is cyclic (or non-derogatory) if there exists a cyclic vector for M , i.e. $v \in \mathbb{C}^n$ such that $(v, Mv, \dots, M^{k-1}v, \dots)$ spans \mathbb{C}^n , which is equivalent to the fact that $(v, Mv, \dots, M^{n-1}v)$ is a basis of \mathbb{C}^n .*

Many equivalent definitions of this notion can be found, for instance in [3] and [4], or [5, Proposition 3]. We point one out: M is cyclic if and only if for any $\lambda \in \mathbb{C}$, $\dim \text{Ker}(M - \lambda I_n) \leq 1$. In particular, any matrix with n distinct eigenvalues is cyclic, and for any given spectrum $\lambda_1, \dots, \lambda_n$, the set of non-cyclic matrices with that spectrum is the algebraic set

$$\{M : \exists j : \dim \text{Ker}(M - \lambda_j I_n) \geq 2\}.$$

Hence the set of non-cyclic matrices is of codimension 1 in the set of matrices which admit at least one multiple eigenvalue, itself of codimension 1 in \mathcal{M}_n .

Theorem 2. *Let F be a spectrum-preserving holomorphic map of Ω_n . Let $X_0 \in \Omega_n$ be a cyclic matrix. Then there exists a neighborhood \mathcal{V}_{X_0} of X_0 and a map Q holomorphic from \mathcal{V}_{X_0} to \mathcal{M}_n^{-1} such that for any $X \in \mathcal{V}_{X_0}$, $F(X) = Q(X)^{-1}XQ(X)$.*

3. PROOF

We fix some notation. For $A \in \mathcal{M}_n$, let

$$\sigma_j(A) := \sigma_j(\lambda_1, \dots, \lambda_n) := (-1)^j \sum_{1 \leq k_1 < \dots < k_j \leq n} \lambda_{k_1} \dots \lambda_{k_j}$$

and $\lambda_1, \dots, \lambda_n$ are the (possibly equal) eigenvalues of A . Those are polynomials in the coefficients of A . Put $\sigma := (\sigma_1, \dots, \sigma_n) : \mathcal{M}_n \rightarrow \mathbb{C}^n$.

Conversely, given $a := (a_1, \dots, a_n) \in \mathbb{C}^n$, the associated *companion matrix* \mathcal{C}_a is

$$\begin{pmatrix} 0 & & & -a_n \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}.$$

The companion matrix associated to a matrix A is $\mathcal{C}_{\sigma(A)}$. They have the same characteristic polynomial, or equivalently $\sigma(\mathcal{C}_{\sigma(A)}) = \sigma(A)$.

Now given a matrix X_0 as in the Theorem, and a vector v_0 cyclic for X_0 , let

$$\mathcal{U}_{X_0} := \{M \in \mathcal{M}_n : \det(v_0, Mv_0, \dots, M^{n-1}v_0) \neq 0\}.$$

This is a neighborhood of X_0 . Let $P_{v_0}(M)$ be the matrix with columns $(v_0, Mv_0, \dots, M^{n-1}v_0)$; this depends polynomially on the entries of M , and is invertible. One can see that for $X \in \mathcal{U}_{X_0}$,

$$P_{v_0}(X)^{-1}XP_{v_0}(X) = \mathcal{C}_{\sigma(X)}$$

(the $n-1$ st columns coincide, and they have the same characteristic polynomial).

By the Baribeau-Ransford theorem [1] $F(X_0)$ is conjugate to X_0 , therefore cyclic. So there is a neighborhood $\mathcal{U}_{F(X_0)}$ where the relation $P_{w_0}(Y)^{-1}YP_{w_0}(Y) = \mathcal{C}_{\sigma(Y)}$ holds. Take $\mathcal{V}_{X_0} \subset \mathcal{U}_{X_0}$ small enough so that $F(\mathcal{V}_{X_0}) \subset \mathcal{U}_{F(X_0)}$. For any $X \in \mathcal{V}_{X_0}$, using the fact that F is spectrum-preserving,

$$\begin{aligned} F(X) &= P_{w_0}(F(X))\mathcal{C}_{\sigma(F(X))}P_{w_0}(F(X))^{-1} \\ &= P_{w_0}(F(X))\mathcal{C}_{\sigma(X)}P_{w_0}(F(X))^{-1} \\ &= P_{w_0}(F(X))P_{v_0}(X)^{-1}XP_{v_0}(X)P_{w_0}(F(X))^{-1}, \end{aligned}$$

so that we have the theorem with $Q(X) = P_{v_0}(X)P_{w_0}(F(X))^{-1}$. \square

Notice that there is no hope to make a global holomorphic choice of v_0 on the whole open of cyclic matrices. Indeed, since the complement of this is of codimension 2, we could then extend it to the whole of

Ω_n by Hartog's phenomenon, but it would mean that all matrices are cyclic, which is obviously false.

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